Identities for the inverses of central binomial coefficients

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Abstract. The main aim of this paper is a discussion on the convolution-type identity for the inverses of central binomial coefficients. Moreover, many new identities of algebraic and trigonometric nature for these rational numbers are obtained, which complete the ones derived by the authors in their previous papers [10, 12] and by Mattarei and Tauraso in [3]. Two new triangles of the rational numbers are also discovered.

Keywords: central binomial coefficient, generating function for inverses of central binomial coefficients, convolution formulae for inverses of central binomial coefficients, Chebyshev polynomials of the first and second kind, Parker's identity.

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1. Introduction

This paper is a follow-up of the research initiated by the authors in [10], inspired, first and foremost, by Lehmer's publication [2], where many attractive results have implied new questions. By posing new problems, associated, in the majority of cases, with the forms of convolution-type sums of the inverses of central binomial coefficients, the authors have drafted this paper. Additional incentives for the paper were our previous publications, namely [12] and [13].

Our paper is divided into four sections. In the second one, using some Lehmer's formulae from [2], we generate our two fundamental identities for the self convolution of sequence of the inverses of central binomial coefficients. In the next section the new inspiring trigonometric identities connected with the inverses of central binomial coefficients are shown. On that base two triangles of the rational numbers are created

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and next the new algebraic identities for the inverses of central binomial coefficients are derived. At the end of the paper, by using the almost forgotten (and discovered again independently by us) Parker's identity and some Ziad S. Ali identity, the final form of formula for the self convolution of sequence of the inverses of central binomial coefficients is presented – see formula (45) in frame.

2. Convolution-type identities

At first, our aim is to generate the following convolution-type identities for the inverses of central binomial coefficients

$$a_n := \binom{2n}{n},$$

for every $n = 0, 1, 2, \ldots$

Theorem 2.1. *The following identities hold*

$$4^n \sum_{k=1}^{n-1} \frac{1}{a_k\, a_{n-k}} = n - 1 + \sum_{k=1}^{n-1} (n + 3\,k + 1)\,(n-k)\,\frac{4^{k-1}}{k^2\, a_k} \tag{1}$$

and

$$\begin{aligned} 4^n \sum_{k=1}^{n-1} \frac{1}{a_k\, a_{n-k}} &= n - \frac{1}{4} - \frac{(n+1)\,4^{n-1}}{n\, a_n} + (2\,n-1) \sum_{k=1}^{n-1} \frac{4^{k-1}}{k\, a_k} \\ &\quad - \frac{1}{2\, a_n} \sum_{k=1}^{n} \binom{2\,n}{n+k} \left(4\,k^2 - 1 + n\,(n+1)\,\frac{(-1)^k - 1}{k^2}\right), \end{aligned} \tag{2}$$

for every $n = 2, 3, \ldots$

Proof. From the following Lehmer's formula (see [2, formula (15)]):

$$\sum_{n=1}^{\infty} \frac{(2\,x)^{2n}}{a_n} = \frac{x^2}{1-x^2} + \frac{x\,\arcsin(x)}{(1-x^2)^{3/2}}, \tag{3}$$

we easily obtain the formula

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \frac{(2\,x)^{2n}}{a_n}\right)^2 &= x^4\,(1-x^2)^{-2} + 2\,x^3\,(1-x^2)^{-5/2}\,\arcsin(x) \\ &\quad + x^2\,(1-x^2)^{-3}\,\big(\arcsin(x)\big)^2. \end{aligned} \tag{4}$$

But, by the binomial series, we have

$$(1-x^2)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} (-x^2)^n = \sum_{n=0}^{\infty} (n+1)\,x^{2n}. \tag{5}$$

Hence, by the binomial series and by formula (9) from [2], we deduce

$$
\begin{aligned}
2\,x^3\,(1-x^2)^{-5/2}\,\arcsin(x) &= x^2\,(1-x^2)^{-2}\,\frac{2\,x}{\sqrt{1-x^2}}\,\arcsin(x)\\
&= x^2\left(\sum_{n=0}^{\infty}(n+1)\,x^{2n}\right)\left(\sum_{n=1}^{\infty}\frac{(2\,x)^{2n}}{n\,a_n}\right)\\
&= x^2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(n-k+1)\,\frac{2^{2k}}{k\,a_k}\right)x^{2n};
\end{aligned}
\tag{6}
$$

and, at last, by the binomial series and by formula (13) from [2], we get

$$
\begin{aligned}
\frac{x^2}{(1-x^2)^3}\big(\arcsin(x)\big)^2 &= \frac{x^2}{2}\left(\sum_{n=0}^{\infty}\binom{n+2}{2}x^{2n}\right)\left(\sum_{n=1}^{\infty}\frac{(2\,x)^{2n}}{n^2\,a_n}\right)\\
&= \frac{x^2}{2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\binom{n-k+2}{2}\frac{2^{2k}}{k^2\,a_k}\right)x^{2n}.
\end{aligned}
\tag{7}
$$

On the other hand, by applying the Cauchy product, we get

$$
\left(\sum_{n=1}^{\infty}\frac{(2\,x)^{2n}}{a_n}\right)^2=\sum_{n=2}^{\infty}\left(4^n\sum_{k=1}^{n-1}\frac{1}{a_k\,a_{n-k}}\right)x^{2n},
$$

which, by formulae (4)–(7) and (3.6) from paper [12], yields the following formula

$$
\begin{aligned}
4^n\sum_{k=1}^{n-1}\frac{1}{a_k\,a_{n-k}} &= n-1+\sum_{k=1}^{n-1}(n+3\,k+1)\,(n-k)\,\frac{4^{k-1}}{k^2\,a_k}\\
&= n-\frac{1}{4}+n\,(n+1)\sum_{k=1}^{n-1}\frac{4^{k-1}}{k^2\,a_k}\\
&\quad+(2\,n-1)\sum_{k=1}^{n-1}\frac{4^{k-1}}{k\,a_k}-\frac{1}{2\,a_n}\sum_{k=1}^{n}\binom{2\,n}{n+k}(4\,k^2-1),
\end{aligned}
\tag{8}
$$

which, by formula (5.1) from [12] for $\varphi=0$, implies (2). □

Let us state that identity (2) is not in final form in view of the necessity of deriving a reduction formula for the following sum

$$
\sum_{k=1}^{n}\frac{4^k}{k\,a_k}.
\tag{9}
$$

The next results will be focused on this problem. The corresponding trigonometric type reduction formula obeying sum (9) is given by identity (32). It is derived as a special case of the more general identities of trigonometric nature for the inverses of central binomial numbers (formulae (10), (16) and (29)). All of them follow from the formulae obtained by the authors in [12]. Simultaneously, they constitute the nontrivial complement of identities derived in [12]. At last the identity (34) is a compact algebraic form of the sum (9).

3. Auxiliary functions $\mathcal{A}_k(\varphi)$

Now we discuss some trigonometric type identities connected with the inverses of central binomial coefficients. By differentiating formula (1.1) from [12] divided by $\sin\varphi$, we obtain as follows.

Theorem 3.1. *For every $n \in \mathbb{N}$, $n \geq 2$ the following identity holds*

$$a_n \left(\tfrac{1}{2}\, \tan(\varphi) + \sum_{k=1}^{n-1} a_k^{-1}\, \sin(\varphi)\, \big(2\, \cos(\varphi)\big)^{2k-1}\right) = \sum_{k=1}^{n} \binom{2\,n}{n+k} \mathcal{A}_k(\varphi), \tag{10}$$

where

$$\mathcal{A}_k(\varphi) := \frac{\cos(\varphi)\, \sin(2\,k\,\varphi) - 2\,k\, \cos(2\,k\,\varphi)\, \sin(\varphi)}{2\,k\, \cos(\varphi)\, \sin^2(\varphi)}, \qquad k \in \mathbb{N},$$

whence the functions $\mathcal{A}_k(\varphi)$ do not depend on n.

Moreover, the following recurrence formula holds

$$\mathcal{A}_{k+1}(\varphi) = \frac{k}{k+1}\, \cos(2\,\varphi)\, \mathcal{A}_k(\varphi) + \frac{1}{k+1}\, \tan(\varphi)\, \cos(2\,k\,\varphi) + 2\, \sin(2\,k\,\varphi), \tag{11}$$

for every $k \in \mathbb{N}$.

Proof. We have

$$\begin{aligned}
\mathcal{A}_{k+1}(\varphi) &= \big((k+1)\, \sin(\varphi)\, \sin(2\,\varphi)\big)^{-1} \big(\cos(\varphi)\, \sin(2\,k\,\varphi + 2\,\varphi) \\
&\qquad - (2\,k+2)\, \cos(2\,k\,\varphi + 2\,\varphi)\, \sin(\varphi)\big) \\
&= \big((k+1)\, \sin(\varphi)\, \sin(2\,\varphi)\big)^{-1} \big(\cos(\varphi)\, \sin(2\,k\,\varphi)\, \cos(2\,\varphi) \\
&\quad + \cos(\varphi)\, \cos(2\,k\,\varphi)\, \sin(2\,\varphi) - (2\,k+2)\, \cos(2\,k\,\varphi)\, \cos(2\,\varphi)\, \sin(\varphi) \\
&\qquad + (2\,k+2)\, \sin(2\,k\,\varphi)\, \sin(2\,\varphi)\, \sin(\varphi)\big) \\
&= \big((k+1)\, \sin(\varphi)\, \sin(2\,\varphi)\big)^{-1} \big(\cos(2\,\varphi)\, \big(\cos(\varphi)\, \sin(2\,k\,\varphi) \\
&\qquad - 2\,k\, \cos(2\,k\,\varphi)\, \sin(\varphi)\big) + \cos(\varphi)\, \cos(2\,k\,\varphi)\, \sin(2\,\varphi) \\
&\qquad - 2\, \cos(2\,k\,\varphi)\, \cos(2\,\varphi)\, \sin(\varphi)\big) + 2\, \sin(2\,k\,\varphi) \\
&= \frac{k}{k+1}\, \cos(2\,\varphi)\, \mathcal{A}_k(\varphi) + \big((k+1)\, \sin(\varphi)\, \sin(2\,\varphi)\big)^{-1} \cos(2\,k\,\varphi) \times \\
&\qquad \times \big(\cos(\varphi)\, \sin(2\,\varphi) - 2\, \cos(2\,\varphi)\, \sin(\varphi)\big) + 2\, \sin(2\,k\,\varphi) \\
&= \frac{k}{k+1}\, \cos(2\,\varphi)\, \mathcal{A}_k(\varphi) + \frac{1}{k+1}\, \cos(2\,k\,\varphi)\, \mathcal{A}_1(\varphi) + 2\, \sin(2\,k\,\varphi) \\
&= \frac{k}{k+1}\, \cos(2\,\varphi)\, \mathcal{A}_k(\varphi) + \frac{1}{k+1}\, \tan(\varphi)\, \cos(2\,k\,\varphi) + 2\, \sin(2\,k\,\varphi),
\end{aligned}$$

which implies (11). □

We note that

$$\mathcal{A}_1(\varphi) = \frac{\cos^2(\varphi) - \cos(2\,\varphi)}{\cos(\varphi)\,\sin(\varphi)} = \tan(\varphi), \tag{12}$$

$$\begin{aligned}
\mathcal{A}_2(\varphi) &= \frac{\cos^2(\varphi)\,\cos(2\,\varphi) - \cos(4\,\varphi)}{\cos(\varphi)\,\sin(\varphi)} \\
&= \frac{\big(\cos^2(\varphi)\,\cos(2\,\varphi) - \cos^2(2\,\varphi)\big) + \sin^2(2\,\varphi)}{\cos(\varphi)\,\sin(\varphi)} \\
&= \frac{\cos(2\,\varphi)\,\big(\cos^2(\varphi) - \cos(2\,\varphi)\big)}{\cos(\varphi)\,\sin(\varphi)} + 2\,\sin(2\,\varphi) \\
&= \cos(2\,\varphi)\,\tan(\varphi) + 2\,\sin(2\,\varphi) = \big(2\,\cos^2(\varphi) - 1\big)\,\tan(\varphi) + 2\,\sin(2\,\varphi) \\
&= 3\,\sin(2\,\varphi) - \tan(\varphi),
\end{aligned} \tag{13}$$

and from (11) we obtain

$$\begin{aligned}
\mathcal{A}_3(\varphi) &= \frac{2}{3}\,\cos(2\,\varphi)\,\big(3\,\sin(2\,\varphi) - \tan(\varphi)\big) + \frac{1}{3}\,\tan(\varphi)\cos(4\,\varphi) + 2\,\sin(4\,\varphi) \\
&= 3\,\sin(4\,\varphi) + \frac{1}{3}\,\tan(\varphi)\,\big(\cos(4\,\varphi) - 2\,\cos(2\,\varphi)\big) \\
&= \frac{10}{3}\,\sin(4\,\varphi) - \frac{4}{3}\,\sin(2\,\varphi) + \tan(\varphi),
\end{aligned} \tag{14}$$

since

$$\begin{aligned}
\cos(4\varphi) - 2\,\cos(2\varphi) &= T_4(\cos(\varphi)) - 2T_2(\cos(\varphi)) \\
&= 8\cos^4(\varphi) - 12\cos^2(\varphi) + 3 = \cos(\varphi)U_3(\cos(\varphi)) - 8\cos^2(\varphi) + 3 \\
&= \cot(\varphi)\sin(4\varphi) - 8\cos^2(\varphi) + 3,
\end{aligned}$$

where $T_2(x)$, $T_4(x)$ and $U_3(x)$ denote the respective Chebyshev polynomials of the first and second kind. Some of the above relations can be derived from the following auxiliary Lemma.

Lemma 3.2. *The following identity holds*

$$\begin{aligned}
\tan(\varphi)\,T_{2n}\big(\cos(\varphi)\big) &= \sin(2\,n\,\varphi) - 2\,\sin(2\,(n-1)\,\varphi) + \dots \\
&\dots + 2\,(-1)^{n-1}\,\sin(2\,\varphi) + (-1)^n\,\tan(\varphi),
\end{aligned} \tag{15}$$

where $T_n(x)$ denotes the n-th Chebyshev polynomial of the first kind [4, 7].

Proof. By means of the basic identities for $U_n(x)$ (see [4, 7]), where $U_n(x)$ denotes the n-th Chebyshev polynomial of the second kind, we obtain

$$\begin{aligned}\tan(\varphi)\, T_{2n}\big(\cos(\varphi)\big) &= \tan(\varphi)\,\Big(\cos(\varphi)\, U_{2n-1}\big(\cos(\varphi)\big) - U_{2n-2}\big(\cos(\varphi)\big)\Big)\\ &= \sin(2\,n\,\varphi) - \tan(\varphi)\, U_{2n-2}\big(\cos(\varphi)\big)\\ &= \sin(2\,n\,\varphi) - \tan(\varphi)\,\Big(2\,\cos(\varphi)\, U_{2n-3}\big(\cos(\varphi)\big) - U_{2n-4}\big(\cos(\varphi)\big)\Big)\\ &= \sin(2\,n\,\varphi) - 2\,\sin(2\,(n-1)\,\varphi) + 2\,\sin(2\,(n-2)\,\varphi) + \ldots\\ &\qquad \ldots + 2\,(-1)^{n-1}\,\sin(2\,\varphi) + (-1)^n\,\tan(\varphi).\end{aligned}$$

□

Now we present the generalization of formulae (12), (13) and (14).

Theorem 3.3. *The following formula holds*

$$\mathcal{A}_k(\varphi) := \frac{1}{k}\,\left(\sum_{l=1}^{k-1}(-1)^{k-1-l}\,a_{l,k}\,\sin(2\,l\,\varphi)\right) + (-1)^{k-1}\,\tan(\varphi), \tag{16}$$

where

$$a_{k,k+1} = \tfrac{1}{2}\,a_{k-1,k} + 2\,k + 3, \tag{17}$$
$$a_{k-1,k+1} = \tfrac{1}{2}\,a_{k-2,k} + 2, \tag{18}$$
$$a_{k-2,k+1} = \tfrac{1}{2}\,(a_{k-1,k} + a_{k-3,k}) + 2, \tag{19}$$
$$\vdots$$
$$a_{i,k+1} = \tfrac{1}{2}\,(a_{i+1,k} + a_{i-1,k}) + 2, \qquad i = 2, 3, \ldots, k-2, \tag{20}$$
$$\vdots$$
$$a_{1,k+1} = \tfrac{1}{2}\,a_{2,k} + k + 2, \tag{21}$$

and

$$\begin{array}{llll} a_{1,2} = 6, & a_{1,4} = 10, & a_{1,5} = 8, & a_{1,6} = 14,\\ a_{1,3} = 4, & a_{2,4} = 4, & a_{2,5} = 14, & a_{2,6} = 8,\\ a_{2,3} = 10, & a_{3,4} = 14, & a_{3,5} = 4, & a_{3,6} = 18,\\ & & a_{4,5} = 18, & a_{4,6} = 4,\\ & & & a_{5,6} = 22. \end{array} \tag{22}$$

Proof. We proceed by induction on k. It results from (12)–(14) the identity (16) holds for $k = 1, 2, 3$. Now suppose that (16) holds for some $k \in \mathbb{N}$. Then by recurrence formula (11) and Lemma 3.2, we get

$$
\begin{aligned}
\mathcal{A}_{k+1}(\varphi) &= \frac{k}{k+1}\,\cos(2\varphi)\,\mathcal{A}_k(\varphi) + \frac{1}{k+1}\,\tan(\varphi)\,\cos(2k\varphi) + 2\sin(2k\varphi) \\
&= \frac{1}{k+1}\Big(\frac{1}{2}\sum_{l=1}^{k-1}(-1)^{k-1-l}a_{l,k}\big(\sin\big(2\,(l+1)\,\varphi\big)+\sin\big(2\,(l-1)\,\varphi\big)\big)\Big) \\
&+ \frac{k}{k+1}\,(-1)^{k-1}\,\big(\sin(2\,\varphi)-\tan(\varphi)\big) + \frac{1}{k+1}\,\sin(2\,k\,\varphi) - \frac{2}{k+1}\,\sin(2\,(k-1)\,\varphi) \\
&\quad \ldots + \frac{2}{k+1}\,(-1)^{k-1}\,\sin(2\,\varphi) + (-1)^k\,\frac{1}{k+1}\,\tan(\varphi) + 2\,\sin(2\,k\,\varphi) \\
&= \frac{1}{k+1}\,\Big(\big(\tfrac{1}{2}\,a_{k-1,k}+2\,k+3\big)\,\sin(2\,k\,\varphi) - \big(\tfrac{1}{2}\,a_{k-2,k}+2\big)\,\sin(2\,(k-1)\,\varphi) \\
&\qquad + \big(\tfrac{1}{2}\,(a_{k-1,k}+a_{k-3,k})+2\big)\,\sin(2\,(k-2)\,\varphi) \\
&\qquad - \big(\tfrac{1}{2}\,(a_{k-2,k}+a_{k-4,k})+2\big)\,\sin(2\,(k-3)\,\varphi) \\
&\qquad \ldots + (-1)^{k-2}\,\big(\tfrac{1}{2}\,(a_{3,k}+a_{1,k})+2\big)\,\sin(4\,\varphi) \\
&\qquad\quad + (-1)^{k-1}\,\big(\tfrac{1}{2}\,a_{2,k}+k+2\big)\,\sin(2\,\varphi)\Big) + (-1)^k\,\tan(\varphi),
\end{aligned}
$$

which implies (17)–(21). □

4. Triangles of numbers $a_{k,n}$ and $b_{k,n}$

In this section we present the explicit form of elements $a_{l,k}$, for every $l, k \in \mathbb{N}$, such that $l < k$ and $k \geq 2$.

Theorem 4.1. *We have*

$$a_{k-2l,k} = 4l, \qquad k = 2l+1, 2l+2, ..., \tag{23}$$

$$a_{k-2l+1,k} = 4(k-l)+2, \qquad k = 2l, 2l+1, 2l+2, ..., \tag{24}$$

for every $l \in \mathbb{N}$.

Hence, we deduce the following equivalent description of elements $a_{l,k}$:

$$a_{k-1,k} = 4k-2, \qquad k = 2,3,4,..., \tag{25}$$

$$a_{k-2,k} = 4, \qquad k = 3,4,5,..., \tag{26}$$

$$a_{k-3,k} = 4k-6, \qquad k = 4,5,6,...,$$

$$a_{k-4,k} = 8, \qquad k = 5,6,7,...,$$

$$\ldots$$

$$a_{1,2l+1} = a_{2,2l+2} = 4l, \qquad l = 1,2,\ldots, \tag{27}$$

$$a_{1,2l} = 4l+2, \quad a_{2,2l+1} = 4l+6, \qquad l = 1,2,\ldots \tag{28}$$

Proof. We will prove inductively that for each $k \in \mathbb{N}$, $k \geq 2$ and for all $\tau = 1,2,...,k-1$ formulae (23) and (24) hold for elements $a_{\tau,k}$. From (22) it follows that only the induction step is needed.

For this purpose let us set $k \in \mathbb{N}$, $k \geq 2$ and let us assume that formulae (23) and (24) describing elements $a_{\tau,k}$, $\tau = 1, 2, ..., k-1$, are true. Thus, by using formulae (17)–(21) we receive, successively:

$$a_{k,k+1} \overset{(25)}{=} \frac{1}{2}(4k-2) + 2k + 3 = 4(k+1) - 2,$$
$$a_{k-1,k+1} \overset{(26)}{=} \frac{1}{2}4 + 2 = 4,$$

if $l \in \mathbb{N}$ and $2 \leq k - 2l + 1 \leq k - 2$ then

$$a_{k-2l+1,k+1} = \frac{1}{2}(a_{k-2l+2,k} + a_{k-2l,k}) + 2 = \frac{1}{2}(4(l-1) + 4l) + 2 = 4l,$$

if $l \in \mathbb{N}$ and $2 \leq k - 2l$ then

$$\begin{aligned} a_{k-2l,k+1} &= \frac{1}{2}(a_{k-2l+1,k} + a_{k-2l-1,k}) + 2 \\ &= \frac{1}{2}(4(k-l) + 2 + 4(k-l-1) + 2) + 2 = 4(k-l) + 2, \end{aligned}$$

and, at last, if $k = 2l + 1$ then

$$a_{1,2l+2} = \frac{1}{2}a_{2,2l+1} + 2l + 3 \overset{(28)}{=} 2l + 3 + 2l + 3 = 4l + 6,$$

whereas, if $k = 2l$ then

$$a_{1,2l+1} = \frac{1}{2}a_{2,2l} + 2l + 2 \overset{(28)}{=} \frac{1}{2}4(l-1) + 2l + 2 = 4l,$$

which finishes the proof. $\square$

Corollary 4.2. *From (23) and (24) (similarly from (25)–(28)) we can generate the following triangle T_1 of values of $a_{k,l}$, $k, l \in \mathbb{N}$, $k < l$:*

6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	4	14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	14	4	18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	8	18	4	22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	18	8	22	4	26	0	0	0	0	0	0	0	0	0	0	0	0	0
18	12	22	8	26	4	30	0	0	0	0	0	0	0	0	0	0	0	0
16	22	12	26	8	30	4	34	0	0	0	0	0	0	0	0	0	0	0
22	16	26	12	30	8	34	4	38	0	0	0	0	0	0	0	0	0	0
20	26	16	30	12	34	8	38	4	42	0	0	0	0	0	0	0	0	0
26	20	30	16	34	12	38	8	42	4	46	0	0	0	0	0	0	0	0
24	30	20	34	16	38	12	42	8	46	4	50	0	0	0	0	0	0	0
30	24	34	20	38	16	42	12	46	8	50	4	54	0	0	0	0	0	0
28	34	24	38	20	42	16	46	12	50	8	54	4	58	0	0	0	0	0
34	28	38	24	42	20	46	16	50	12	54	8	58	4	62	0	0	0	0
32	38	28	42	24	46	20	50	16	54	12	58	8	62	4	66	0	0	0
38	32	42	28	46	24	50	20	54	16	58	12	62	8	66	4	70	0	0
36	42	32	46	28	50	24	54	20	58	16	62	12	66	8	70	4	74	0
42	36	46	32	50	28	54	24	58	20	62	16	66	12	70	8	74	4	78

Finally, we focus on one more trigonometric identity that describes the following sum

$$a_n \sum_{k=1}^{n-1} \Big(k\, a_k\Big)^{-1} \big(2\,\cos(\varphi)\big)^{2k}$$

(see formulae (31) and (32) below). From formulae (10) and (16) we obtain immediately the identity

$$\begin{aligned} a_n \sum_{k=1}^{n-1} a_k^{-1} \sin(\varphi)\,\big(2\,\cos(\varphi)\big)^{2k-1} \\ = \sum_{k=2}^{n} \frac{1}{k} \binom{2\,n}{n+k} \left(\sum_{l=1}^{k-1} (-1)^{k-1-l}\, a_{l,k}\, \sin(2\,l\,\varphi) \right) \\ = \sum_{k=1}^{n-1} b_{k,n}\, \sin(2\,k\,\varphi), \end{aligned} \tag{29}$$

where

$$b_{k,n} := \sum_{l=k+1}^{n} \frac{(-1)^{l-k-1}}{l} \binom{2\,n}{n+l} a_{k,l}, \tag{30}$$

for every $k = 1, 2, \ldots, n-1$. Using the triangle T_1 for numbers $a_{k,l}$ we can generate (with the aid of *Mathematica* software) the next triangle T_2 of values of $b_{k,n}$, $k, n \in \mathbb{N}$, $k < n$:

$$\begin{array}{cccccccccc}
0 & 3 & \frac{50}{3} & \frac{455}{6} & \frac{1617}{5} & \frac{6699}{5} & \frac{191334}{35} & \frac{619047}{28} & \frac{11204479}{126} & \frac{224524729}{630} \\
0 & 0 & \frac{10}{3} & \frac{77}{3} & \frac{714}{5} & \frac{3498}{5} & \frac{112398}{35} & \frac{198627}{14} & \frac{3853135}{63} & \frac{81674749}{315} \\
0 & 0 & 0 & \frac{7}{2} & \frac{171}{5} & \frac{1122}{5} & \frac{43472}{35} & \frac{175461}{28} & \frac{1254175}{42} & \frac{28767349}{210} \\
0 & 0 & 0 & 0 & \frac{18}{5} & \frac{638}{15} & \frac{33748}{105} & \frac{13897}{7} & \frac{689078}{63} & \frac{17650658}{315} \\
0 & 0 & 0 & 0 & 0 & \frac{11}{3} & \frac{1066}{21} & \frac{12155}{28} & \frac{373235}{126} & \frac{2232253}{126} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{26}{7} & \frac{825}{14} & \frac{11815}{21} & \frac{441541}{105} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{4} & \frac{1207}{18} & \frac{63631}{90} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{34}{9} & \frac{3382}{45} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{5} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

From (29) and (30), integrating over φ from $\frac{\pi}{2}$ to φ, we get

$$\begin{aligned} a_n \sum_{k=1}^{n-1} \big(k\, a_k\big)^{-1} \big(2\,\cos(\varphi)\big)^{2k} \\ = \sum_{k=2}^{n} \frac{1}{k} \binom{2\,n}{n+k} \left(\sum_{l=1}^{k-1} \frac{(-1)^{k-1-l}}{2\,l}\, a_{l,k}\, \cos(2\,l\,\varphi) \right) \\ - \sum_{k=2}^{n} \frac{(-1)^{k-1}}{k} \binom{2\,n}{n+k} \left(\sum_{l=1}^{k-1} \frac{1}{2\,l}\, a_{l,k} \right), \end{aligned} \tag{31}$$

i.e.

$$a_n \sum_{k=1}^{n-1} \left(k\, a_k\right)^{-1} \left(2\, \cos(\varphi)\right)^{2k} = \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k}\, b_{k,n}\, \cos(2\, k\, \varphi) - \frac{1}{2} \sum_{k=1}^{n-1} \frac{(-1)^k}{k}\, b_{k,n}. \tag{32}$$

Then, from (32), four special formulae can be derived:

1) for $\varphi = 0$:

$$\begin{aligned} a_n \sum_{k=1}^{n-1} 4^k \left(k\, a_k\right)^{-1} &= \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2\,k-1}\, b_{2k-1,n} \\ &= \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2\,k-1} \sum_{l=2k}^{n} \frac{(-1)^l}{l} \binom{2\,n}{n+l} a_{2k-1,l}; \end{aligned} \tag{33}$$

let us notice that by identity (39) below we have

$$a_n \sum_{k=1}^{n-1} 4^k \left(k\, a_k\right)^{-1} = \left(2 - \frac{1}{n}\right) 4^n - 2a_n \tag{34}$$

so we get

$$\begin{aligned} \left(2 - \frac{1}{n}\right) 4^n - 2a_n &= \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2k-1}\, b_{2k-1,n} \\ &= \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{2\,k-1} \sum_{l=2k}^{n} \frac{(-1)^l}{l} \binom{2n}{n+l} a_{2k-1,l}. \end{aligned} \tag{35}$$

2) for $\varphi = \frac{\pi}{4}$:

$$A_0(n) := a_n \sum_{k=1}^{n-1} 2^k (k\, a_k)^{-1} = \frac{1}{2} \sum_{k=1}^{n-1} \frac{\cos\left(\frac{\pi}{2}k\right) - (-1)^k}{k} b_{k,n}. \tag{36}$$

This formula involves the periodic sequence

$$\left\{\cos\left(\frac{\pi}{2}k\right) - (-1)^k\right\}_{k=1}^{\infty} = \{(1,-2,1,0)\}_{k=1}^{\infty},$$

and the first values of A_0 are equal to $0, \frac{3}{2}, \frac{20}{3}, \frac{77}{3}, 96, \frac{5368}{15}, \frac{140608}{105}, \frac{35272}{7}, \frac{400384}{21}, \frac{22838912}{315}, \frac{45693952}{165}, \frac{525568768}{495}, \frac{404316160}{99}, \frac{15768938496}{1001}, \frac{182923083776}{3003}, \frac{3544166523904}{15015}, \frac{416962576384}{455}, \frac{2362792902656}{663}, \frac{524540560277504}{37791}$;
let us note that $A_0(5) = 96$ is the only integer value in this sequence;

3) for $\varphi = \frac{\pi}{3}$:

$$A_1(n) := a_n \sum_{k=1}^{n-1} (k\, a_k)^{-1} = \frac{1}{2} \sum_{k=1}^{n-1} \frac{\cos\left(\frac{2}{3}\pi k\right) - (-1)^k}{k} b_{k,n}, \tag{37}$$

where we have the periodic sequence

$$\left\{\cos\left(\frac{2\pi}{3}k\right)-(-1)^k\right\}_{k=1}^{\infty}=\left\{\left(\frac{1}{2},-\frac{3}{2},2,-\frac{3}{2},\frac{1}{2},0\right)\right\}_{k=1}^{\infty},$$

and the first values of A_1 are equal to $0, \frac{3}{4}, \frac{35}{12}, \frac{21}{2}, \frac{1521}{40}, \frac{16753}{120}, \frac{5187}{10}, \frac{217869}{112}, \frac{7407665}{1008}, \frac{15638463}{560}, \frac{46915431}{440}, \frac{1079055143}{2640}, \frac{69170205}{44}, \frac{6069685563}{1001}, \frac{201166722109}{8580}, \frac{519680699151}{5720}, \frac{12472336781769}{35360}, \frac{87306357476023}{63648}, \frac{18890849278629}{3536}, \frac{215355681776901}{10336}$; among these values we may see two "nice" equalities: $10A_1(7)=5187$, $44A_1(13)=69170205$;

4) for $\varphi=\frac{\pi}{6}$:

$$A_2(n):=a_n\sum_{k=1}^{n-1}3^k(k\,a_k)^{-1}=\frac{1}{2}\sum_{k=1}^{n-1}\frac{\cos\left(k\frac{\pi}{3}\right)-(-1)^k}{k}b_{k,n}, \tag{38}$$

where we have the periodic sequence

$$\left\{\cos\left(\frac{\pi}{3}k\right)-(-1)^k\right\}_{k=1}^{\infty}=\left\{\left(\frac{3}{2},-\frac{3}{2},0,-\frac{3}{2},\frac{3}{2},0\right)\right\}_{k=1}^{\infty},$$

and the first values of A_2 are equal to $0, \frac{9}{4}, \frac{45}{4}, \frac{189}{4}, \frac{7533}{40}, \frac{29403}{40}, \frac{28431}{10}, \frac{1226907}{112}, \frac{4721733}{112}, \frac{90876411}{560}, \frac{275109291}{440}, \frac{2122752501}{880}, \frac{410095305}{44}, \frac{72208482993}{2002}, \frac{2798672999877}{20020}, \frac{43453641653613}{80080}, \frac{522090046359171}{247520}, \frac{58062235172499}{7072}, \frac{113142667028049}{3536}, \frac{6452214338009277}{51680}$; here we have two attractive equalities: $10A_2(7)=28431$, $44A_2(13)=410095305$.

Final remarks

1° Identities (29)–(33) (as well as the description of coefficients $a_{k,l}$ from Theorem 4.1) constitute a part of the main results of this paper. Let us emphasize that we obtained them by applying the trigonometric methods, among the others by using the Chebyshev polynomials belonging to the most interesting and definitely creative computational methods of the modern number theory and combinatorics.

2° In papers [5] and [11] the following explicit formula (the Parker's formula) was derived

$$1+\sum_{k=1}^{n}\frac{4^k}{2ka_k}=\frac{4^n}{a_n} \tag{39}$$

(without connecting it with the more general functional formulae as it is done in our paper). Hence, on the basis of formula (2) the following convolution-type identity results which can be considered, in some sense, as the clou of one of the main issues in our investigations

$$\begin{aligned}4^n\sum_{k=1}^{n-1}\frac{1}{a_k\,a_{n-k}}&=\frac{1}{4}+(4n-5)\frac{4^{n-1}}{a_n}\\&-\frac{1}{2\,a_n}\sum_{k=1}^{n}\binom{2\,n}{n+k}\Big(4\,k^2-1+n\,(n+1)\,\frac{(-1)^k-1}{k^2}\Big),\end{aligned} \tag{40}$$

where the relation $a_{n-1}=\frac{n}{2(2n-1)}a_n$, $n\in\mathbb{N}$ is used for reducing the expressions.

Moreover, using the *Mathematica* software we got the formula

$$\frac{1}{na_n}\sum_{k=1}^{n}\binom{2n}{n+k}(4k^2-1) = 1+\frac{2n-1}{n+1}\,{}_2F_1(1,1-n,2+n,-1), \tag{41}$$

where the hypergeometric function ${}_2F_1$ is used. The above sum can be also determined by applying the following trigonometric identities (see [14]):

$$\begin{aligned}\sum_{k=1}^{n}\binom{2n}{n+k}\left(\cos k\varphi+(-1)^{k+1}\right) &= 2^{n-1}(1+\cos\varphi)^n\\ &= 2^{2n-1}\cos^{2n}\frac{\varphi}{2},\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{1}{2}a_n+\sum_{k=1}^{n}(-1)^k\binom{2n}{n+k}\cos k\varphi &= 2^{n-1}(1-\cos\varphi)^n\\ &= 2^{2n-1}\sin^{2n}\frac{\varphi}{2},\end{aligned} \tag{43}$$

which implies

$$\sum_{k=1}^{n}\binom{2n}{n+k} \underset{\text{for }\varphi=\pi}{\overset{(43)}{=}} 2^{2n-1}-\frac{1}{2}a_n,$$

$$\sum_{k=1}^{n}\binom{2n}{n+k}k^2 \overset{(42)}{=} -2^{2n-1}\frac{d^2}{d\varphi^2}\cos^{2n}\frac{\varphi}{2}\Big|_{\varphi=0} = -2^{n-1}\frac{d^2}{d\varphi^2}(1+\cos\varphi)^n\Big|_{\varphi=0},$$

and at last the expected equivalent form of formula (41), that is

$$\begin{aligned}&\frac{1}{2a_n}\sum_{k=1}^{n}\binom{2n}{n+k}(4k^2-1)\\ &\quad=\frac{1}{2a_n}\left(-2^{2n+1}\frac{d^2}{d\varphi^2}\cos^{2n}\frac{\varphi}{2}\Big|_{\varphi=0}+\frac{1}{2}a_n-2^{2n-1}\right)\\ &\quad=\frac{1}{4}-\frac{4^{n-1}}{a_n}-\frac{4^n}{a_n}\frac{d^2}{d\varphi^2}\cos^{2n}\frac{\varphi}{2}\Big|_{\varphi=0}=\frac{1}{4}+\frac{4^{n-1}(2n-1)}{a_n},\end{aligned} \tag{44}$$

since

$$\frac{d^2}{d\varphi^2}\cos^{2n}\frac{\varphi}{2}=-\frac{n}{2}.$$

Summing up, formula (40) takes the following compact form, final for our discussions

$$\boxed{a_n\sum_{k=1}^{n-1}\frac{1}{a_k a_{n-k}}=\frac{n}{2}-1-\frac{n(n+1)}{2^{2n+1}}\sum_{k=1}^{n}\binom{2n}{n+k}\frac{(-1)^k-1}{k^2}.} \tag{45}$$

3° Let us notice that by the Stirling formula (see [6, Chapter 3]) from (39) we deduce the following limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{4^k}{2ka_k} = \sqrt{\pi}.$$

Zhi-Wei Sun in [8] proved the amazing identity

$$\sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{ka_k} = \frac{\pi^3}{48}$$

where $H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}$ is the n-th harmonic number of the second order for every $n \in \mathbb{N}$ and $H_0^{(2)} := 0$ and at last, in [1] it is proved that

$$2n \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} H_{2k} = \frac{4^n}{a_n} + 1$$

for every $n \in \mathbb{N}$, where $H_n := \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}$.

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